Correlation functions of one-dimensional Lieb–Liniger anyons

2007 J. Phys. A: Math. Theor. 40 14963

(http://iopscience.iop.org/1751-8121/40/50/004)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 14963–14984 doi:10.1088/1751-8113/40/50/004

Correlation functions of one-dimensional Lieb–Liniger anyons

Ovidiu I Pâţu[1,3], **Vladimir E Korepin**[1] **and Dmitri V Averin**[2]

[1] CN Yang Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, NY 11794-3840, USA
[2] Department of Physics and Astronomy, State University of New York at Stony Brook, Stony Brook, NY 11794-3800, USA

E-mail: ipatu@grad.physics.sunysb.edu, korepin@max2.physics.sunysb.edu and dmitri.averin@stonybrook.edu

Received 30 July 2007
Published 28 November 2007
Online at stacks.iop.org/JPhysA/40/14963

Abstract

We have investigated the properties of a model of 1D anyons interacting through a δ-function repulsive potential. The structure of the quasi-periodic boundary conditions for the anyonic field operators and the many-anyon wavefunctions is clarified. The spectrum of the low-lying excitations including the particle–hole excitations is calculated for periodic and twisted boundary conditions. Using the ideas of the conformal field theory we obtain the large-distance asymptotics of the density and field correlation function at the critical temperature $T = 0$ and at small finite temperatures. Our expression for the field correlation function extends the results in the literature obtained for harmonic quantum anyonic fluids.

PACS numbers: 04.20.Jb, 05.30.Pr

1. Introduction

For hard-core particles moving in two spatial dimensions, one can unambiguously define the notion of braiding of the particle trajectories by introducing the winding number n that gives the number of times the trajectory of one particle encircles another particle. This fact makes it possible to consider 'anyonic' particles with fractional exchange statistics [1, 2], for which the wavefunction acquires the non-trivial phase factor $e^{\pm i2\pi\kappa}$, where κ is the 'statistical parameter', whenever n changes by ±1. This situation can be contrasted with the case of three spatial dimensions where one can define only permutations (no braiding) of point-like particles leading to only integer statistics, i.e. $\kappa = 0, 1$ for bosons and fermions, respectively. In physical terms, the anyons in two dimensions can be viewed as the charge–flux composites

[3] Permanent address: Institute of Space Sciences MG 23, 077125 Bucharest-Magurele, Romania.

for which the statistical phase arises as the result of the Aharonov–Bohm interaction between the charge of one particle and the flux of the other [3]. Experimentally, anyons can be realized as quasiparticles of the two-dimensional (2D) electron liquids in the fractional quantum Hall effect (FQHE) [4]. Individual quasiparticles are localized and controlled by quantum antidots in the FQHE regime [5], and the transport properties of multi-antidot systems should provide direct manifestations of their fractional exchange statistics [6]. Dynamics of individual FQHE quasiparticles attracted considerable attention (see, e.g., [7, 8]) as a possible basis for realization of the topological quantum computation [9].

Both conceptually and in practice (e.g., in FQHE systems), the 2D anyons can be confined to move in one dimension. There are, however, the aspects of fractional statistics in one dimension that make its introduction more complicated than in two dimensions. One is that for strictly 1D particles, a trajectory of one particle cannot wind around another, making the sign of the exchange phase $e^{\pm i\pi\kappa\,\mathrm{sgn}(x_i-x_j)/2}$ that the wavefunction should acquire when the particle with coordinate x_i moves past the one with x_j, undetermined. The sign of this phase depends on whether x_i rotates clockwise or counter clockwise around x_j in the underlying 2D geometry, which also explains why the signs of the phase change at $x_i = x_j$ are opposite for the two particles in the pair: rotation of one sense for increasing coordinate x_i implies the opposite rotation for increasing x_j. This fact hindered the early attempts at direct introduction of the 1D anyons as charge–flux composites [10, 11]. It implies that any description of the 1D anyons requires an additional convention on the choice of the sign of the statistical phase for each pair of particles. As discussed in more details below, this choice can be arbitrary and affects the appropriate boundary conditions of the quantum-mechanical wavefunctions of the system of anyons.

Another complicating aspect of the fractional statistics in 1D is the interplay between the two types of statistics, exchange statistics discussed in the preceding paragraph and the exclusion statistics defined through the volume of the phase-space occupied by one particle [12]. The exclusion statistics provides effective description of the dynamic interaction of particles, while the exchange statistics is associated with the 'real' non-thermodynamic statistical effects that continue to exist in the limit of hard-core particles with infinite repulsion. The model of 1D anyons with δ-function interaction considered in this work contains both types of effects, and the interplay between them can be seen in equation (25) for the renormalization of the dynamic particle–particle interaction by the exchange statistics. The renormalized interaction determines the thermodynamics of the model and can be expressed in terms of the exclusion statistics. However, in the hard-core limit $c \to \infty$, effective interaction constant is essentially independent of the exchange statistics, and the main features of the thermodynamics of the model coincide with that of free fermions. There are still anyonic effects (e.g., the shift of the quasiparticle momenta by the parameter of the exchange statistics κ) in this limit.

The purpose of this work is to provide a systematic description of the ground state, low-lying excitations and the asymptotics of the correlation function of the gas of 1D anyons with the δ-function repulsion. In the form used below, the model was introduced by Kundu [13], who also provided the Bethe-Ansatz solution. It was further analyzed recently by Batchelor *et al* [14–16]. The model is an anyonic extension of the Bose gas with δ-function interaction solved by Lieb and Liniger [17]. The anyon gas reduces to the Bose gas in the limit when the statistical parameter κ vanishes. In general, the Bethe equations for anyons are equivalent to the Bethe equations for bosons with two effects of the statistics κ: renormalized coupling constant, and a twist in the boundary conditions. This makes it possible for us to use some of the known results for the Bose gas in the discussion of anyons. (Detailed description of the Bose gas with δ-function interaction including the correlation functions can be found in [18].) The main results of our work are the formulae (77), (82) for the density correlation function and (87), (92) for the field correlation function. Expressions for the field correlators extend the results of Calabrese and Mintchev [19] obtained in the harmonic fluid approach by including the higher-order terms that correspond to particle–hole excitations. Also, the conformal field theory approach we use provides immediate generalization of the zero-temperature correlators to finite temperatures.

There are other 1D models of anyons in the literature. Liguori, Mintchev and Pilo [20] investigated the momentum distribution of a more general gas of free anyons and predicted anyon condensation in a certain range of the statistical parameter. Ilieva and Thirring [21] studied the Hilbert space structure of the anyonic field, and showed that for a fixed statistical parameter it can be represented as an orthogonal sum of sectors with different numbers of particles. The Hilbert space of our model has the same structure.

The paper is organized as follows. Sections 2 introduces the field theoretical model of the 1D gas of anyons with periodic and twisted boundary conditions. Equivalent quantum-mechanical problem is formulated in section 3. In section 4, we discuss the properties of the ground state, and in section 5 calculate the finite-size corrections for the ground state and properties of the low-lying excitations. In section 6, using the conformal field theory approach we find the large-distance asymptotics for the zero-temperature density and field correlation functions and correlators at small finite temperatures. Appendix A presents the discussion of the boundary conditions for many-anyonic wavefunctions used in section 3. Appendix B describes the calculation of the energy and momentum of particle–hole excitations for periodic and twisted boundary conditions.

2. The Lieb–Liniger gas of anyons

We consider a gas of anyons with δ-function interaction in one dimension characterized by the Hamiltonian

$$H = \int_0^L \mathrm{d}x \left\{ \left[\partial_x \Psi_A^\dagger(x)\right]\left[\partial_x \Psi_A(x)\right] + c\Psi_A^\dagger(x)\Psi_A^\dagger(x)\Psi_A(x)\Psi_A(x) \right\}, \tag{1}$$

where $c > 0$ is the coupling constant and L the length of the system. The anyonic fields obey the equal-time commutation relations

$$\Psi_A(x_1)\Psi_A^\dagger(x_2) = \mathrm{e}^{-\mathrm{i}\pi\kappa\epsilon(x_1-x_2)}\Psi_A^\dagger(x_2)\Psi_A(x_1) + \delta(x_1 - x_2), \tag{2}$$

$$\Psi_A^\dagger(x_1)\Psi_A^\dagger(x_2) = \mathrm{e}^{\mathrm{i}\pi\kappa\epsilon(x_1-x_2)}\Psi_A^\dagger(x_2)\Psi_A^\dagger(x_1), \tag{3}$$

$$\Psi_A(x_1)\Psi_A(x_2) = \mathrm{e}^{\mathrm{i}\pi\kappa\epsilon(x_1-x_2)}\Psi_A(x_2)\Psi_A(x_1), \tag{4}$$

where

$$\epsilon(x_1 - x_2) = \begin{cases} 1 & \text{when} \quad x_1 > x_2, \\ -1 & \text{when} \quad x_1 < x_2, \\ 0 & \text{when} \quad x_1 = x_2. \end{cases} \tag{5}$$

In the original work [13] introducing this model, the anyonic fields were realized in terms of the bosonic fields

$$\Psi_A^\dagger(x) = \Psi_B^\dagger(x)\,\mathrm{e}^{\mathrm{i}\pi\kappa\int_0^x \mathrm{d}x'\rho(x')}, \qquad \Psi_A(x) = \mathrm{e}^{-\mathrm{i}\pi\kappa\int_0^x \mathrm{d}x'\rho(x')}\Psi_B(x), \tag{6}$$

where

$$\rho(x) \equiv \Psi_A^\dagger(x)\Psi_A(x) = \Psi_B^\dagger(x)\Psi_B(x). \tag{7}$$

Due to the fact that at coinciding points $\epsilon(0) = 0$, the commutation relations (2)–(4) are indeed bosonic. An alternative realization in terms of the fermionic fields was proposed in [22]. However, in this case, the interaction term in the Hamiltonian (1) vanishes, since $\Psi_A^2(x) = \left[\Psi_A^\dagger(x)\right]^2 = 0$ in coinciding points (see also the discussion in [16]). One implication of this difference is that in comparison to the bosonic representation (6), similar fermionic representation with appropriate modification of the statistical parameter, effectively makes it possible to describe only the infinite repulsion limit $c \to \infty$.

Characteristics of the anyonic gas (1) depend on the boundary conditions imposed on the system at $x = 0 = L$. In this work, we use two different quasiperiodic boundary conditions which impose periodicity either directly on the anyonic or on the bosonic fields. Equations (6) imply that the periodic boundary condition for anyons correspond to twisted boundary conditions for bosons and viceversa. In terms of the anyonic fields, the boundary condition we use are

$$\text{periodic BC:} \quad \Psi_A^\dagger(0) = \Psi_A^\dagger(L) \tag{8}$$

and

$$\text{twisted BC:} \quad \Psi_A^\dagger(0) = \Psi_A^\dagger(L)\,\mathrm{e}^{-\mathrm{i}\pi\kappa(N-1)}, \tag{9}$$

where N is the number of particle in the system. One can see directly from equation (6) that the external phase shift $\pi\kappa(N-1)$ introduced into the conditions (9), ensures the periodicity of the bosonic fields. As will be shown in more details below, this means that this phase removes the anyonic shift of the quasiparticle momenta. Below, we use the common notation for the two types of boundary conditions:

$$\Psi_A^\dagger(0) = \Psi_A^\dagger(L)\,\mathrm{e}^{-\mathrm{i}\pi\beta\kappa(N-1)}, \qquad \beta = 0, 1. \tag{10}$$

An important difference of the anyons with fractional exchange statistics from the integer-statistics particles is that the boundary conditions (10) for the fields do not translate directly into the same boundary conditions for the quantum-mechanical wavefunctions of the N-anyon system [6], which have more complicated structure (23) derived in appendix A.

The corresponding equation of motion $-\mathrm{i}\partial_t\Psi_A(x,t) = [H, \Psi_A(x,t)]$ for the boundary conditions (10) is the nonlinear Schrödinger equation

$$\mathrm{i}\partial_t\Psi_A(x,t) = \partial_x\Psi_A(x,t) + 2c\Psi_A^\dagger(x,t)\Psi_A^2(x,t). \tag{11}$$

The number of particle operator Q and the momentum operator P are defined as

$$Q = \int_0^L \mathrm{d}x\, \Psi_A^\dagger(x)\Psi_A(x), \tag{12}$$

$$P = -\frac{\mathrm{i}}{2}\int_0^L \mathrm{d}x\left(\Psi_A^\dagger(x)\partial_x\Psi_A(x) - \left[\partial_x\Psi_A^\dagger(x)\right]\Psi_A(x)\right). \tag{13}$$

Both of them are Hermitian operators which commute with the Hamiltonian

$$[H, P] = [H, Q] = 0. \tag{14}$$

If we define the Fock vacuum as

$$\Psi_A(x)|0\rangle, \qquad x \in [0, L], \tag{15}$$

the N-particle eigenstate of the Hamiltonian (and also of P and Q) can be then written as

$$|\psi\rangle_N = \int \mathrm{d}^N x\, \mathrm{e}^{-\frac{\mathrm{i}\pi\kappa N}{2}} \chi_N(x_1, \ldots, x_N)\Psi_A^\dagger(x_1)\cdots\Psi_A^\dagger(x_N)|0\rangle, \tag{16}$$

where the many-body wavefunction obeys

$$\chi_N(x_1, \ldots, x_i, x_{i+1}, \ldots, x_N) = \mathrm{e}^{-\mathrm{i}\pi\kappa\epsilon(x_i - x_{i+1})}\chi_N(x_1, \ldots, x_{i+1}, x_i, \ldots, x_N). \tag{17}$$

This can be seen directly by using the exchange relation of the field operators $\Psi_A^\dagger(x_i)\Psi_A^\dagger(x_{i+1}) = \mathrm{e}^{\mathrm{i}\pi\kappa\epsilon(x_i-x_{i+1})}\Psi_A^\dagger(x_{i+1})\Psi_A^\dagger(x_i)$ and interchanging the name of the integration variables x_i, x_{i+1}. Iterating the exchanges several times we obtain

$$\chi_N(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_N) = \mathrm{e}^{-\mathrm{i}\pi\kappa[\sum_{k=i+1}^{j}\epsilon(x_i-x_k)-\sum_{k=i+1}^{j-1}\epsilon(x_j-x_k)]}\chi_N(x_1,\ldots,x_j,\ldots,x_i,\ldots,x_N). \tag{18}$$

Equation (18) was first obtained in [13].

3. The equivalent quantum-mechanical problem

In [13, 16], it was shown that the eigenvalue problem (for periodic boundary conditions)

$$H|\psi\rangle_N = E_N|\psi\rangle_N, \qquad P|\psi\rangle_N = p_N|\psi\rangle_N \tag{19}$$

can be reduced to the quantum-mechanical problem

$$\mathcal{H}\chi_N = E_N\chi_N, \qquad \mathcal{P}\chi_N = p_N\chi_N, \tag{20}$$

where

$$\mathcal{H}_N = \sum_{j=1}^{N}\left(-\frac{\partial^2}{\partial x_j^2}\right) + 2c\sum_{1\leqslant j\leqslant k\leqslant N}\delta(x_j-x_k), \tag{21}$$

$$\mathcal{P} = \sum_{j=1}^{N}\left(-\frac{\partial}{\partial_{x_j}}\right). \tag{22}$$

These considerations also hold for twisted and all cyclic boundary conditions for field operators. The boundary conditions for the quantum-mechanical wavefunctions of N anyons are (see [6] and appendix A)

$$\begin{aligned}
\chi_N(0,x_2,\ldots,x_N) &= \mathrm{e}^{\mathrm{i}\pi\beta\kappa(N-1)}\chi_N(L,x_2,\ldots,x_N),\\
\chi_N(x_1,0,\ldots,x_N) &= \mathrm{e}^{-\mathrm{i}2\pi\kappa}\,\mathrm{e}^{\mathrm{i}\pi\beta\kappa(N-1)}\chi_N(x_1,L,\ldots,x_N),\\
&\vdots\\
\chi_N(x_1,x_2,\ldots,0) &= \mathrm{e}^{-\mathrm{i}2N\pi\kappa}\,\mathrm{e}^{\mathrm{i}\pi\beta\kappa(N-1)}\chi_N(x_1,x_2\ldots,L),
\end{aligned} \tag{23}$$

where, as defined above, $\beta = 0, 1$ for periodic and twisted boundary conditions (10).

Using the coordinate Bethe Ansatz [13, 14, 16] we can obtain the eigenfunctions of the Hamiltonian (21) as

$$\chi_N = \frac{\mathrm{e}^{-\mathrm{i}\frac{\pi\kappa}{2}\sum_{j<k}\epsilon(x_j-x_k)}}{\sqrt{N!\prod_{j>k}[(\lambda_j-\lambda_k)^2+c'^2]}}\sum_{\mathcal{P}}(-1)^{[\mathcal{P}]}\,\mathrm{e}^{\mathrm{i}\sum_{n=1}^{N}x_n\lambda_{\mathcal{P}_n}}\prod_{j>k}[\lambda_{\mathcal{P}_j}-\lambda_{\mathcal{P}_k}-ic'\epsilon(x_j-x_k)], \tag{24}$$

where ${}^{-1}[\mathcal{P}]$ is the signature of the permutation and

$$c' = \frac{c}{\cos(\pi\kappa/2)} \tag{25}$$

is the coupling constant renormalized by the exchange statistics. The eigenvalues of the Hamiltonian and momentum operators are $E_N = \sum_{j=1}^{N}\lambda_j^2$ and $p_N = \sum_{j=1}^{N}\lambda_j$, respectively. For the boundary conditions (23) we obtain the Bethe equations

$$\mathrm{e}^{\mathrm{i}\lambda_j L} = \mathrm{e}^{\mathrm{i}\pi(1-\beta)\kappa(N-1)}\prod_{k=1,k\neq j}^{N}\left(\frac{\lambda_j-\lambda_k+\mathrm{i}c'}{\lambda_j-\lambda_k-\mathrm{i}c'}\right). \tag{26}$$

The Bethe equations (26) are similar to those obtained by Lieb and Liniger for the Bose gas with repulsive δ-function interaction. In our case, however, the effective coupling constant c' (25) can take negative values. While it can be shown (see, e.g., [18]) that the Bethe roots λ_j are real for $c' > 0$, the roots can become complex for $c' < 0$, and one gets bound states [23]. In this work, we will consider only the case $c' > 0$.

4. Properties of the ground state

Bethe equations (26) can also be written as

$$\lambda_j L + \sum_{k=1}^{N} \theta(\lambda_j - \lambda_k) = 2\pi n_j + \pi\kappa(1-\beta)(N-1), \qquad j = 1, \ldots, N, \tag{27}$$

where

$$\theta(\lambda) = \mathrm{i}\ln\left(\frac{\mathrm{i}c' + \lambda}{\mathrm{i}c' - \lambda}\right), \tag{28}$$

and n_j are integers when N is odd and half-integers when N is even.

4.1. Twisted boundary conditions

In this case ($\beta = 1$), the Bethe equations are similar to those for the Bose gas with periodic boundary conditions [17, 18] with c' as a coupling constant. The ground state is characterized by the set of integers (half-integers) $n_j = j - (N+1)/2$, so the Bethe equations take the form

$$\lambda_j^B L + \sum_{k=1}^{N} \theta\left(\lambda_j^B - \lambda_k^B\right) = 2\pi\left(j - \frac{N+1}{2}\right), \qquad j = 1, \ldots, N. \tag{29}$$

From now on the superscript B will mean that the variables and physical quantities are the same as those for the Bose gas with periodic boundary conditions and coupling constant c'. In the thermodynamic limit $N, L \to \infty$, $D = N/L = \text{const}$, the Bethe roots become dense and fill the symmetric interval $[-q, q]$. The density of roots in this interval obeys the Lieb-Liniger integral equation

$$\rho(\lambda) - \frac{1}{2\pi}\int_{-q}^{q} K(\lambda,\mu)\rho(\mu)\,\mathrm{d}\mu = \frac{1}{2\pi}, \tag{30}$$

where $K(\lambda,\mu) = \theta'(\lambda-\mu) = 2c'/(c'^2 + (\lambda-\mu)^2)$. The Fermi momentum q can be obtained from the Lieb–Liniger integral equation and the particle density is

$$D = \frac{N}{L} = \int_{-q}^{q} \rho(\lambda)\,\mathrm{d}\lambda. \tag{31}$$

Finally, the energy and the momentum of the ground state are

$$E_0^B = L\int_{-q}^{q} \lambda^2 \rho(\lambda)\,\mathrm{d}\lambda, \qquad P_0^B = 0. \tag{32}$$

4.2. Periodic boundary conditions

This is the case treated in [14–16]. The Bethe equations (27) in this case ($\beta = 0$) are similar to those for the Bose gas with twisted boundary conditions:

$$\lambda_j L + \sum_{k=1}^{N} \theta(\lambda_j - \lambda_k) = 2\pi n_j + \pi\kappa(N-1), \qquad j = 1, \ldots, N. \tag{33}$$

Introducing the notation $\{[\cdots]\}$ such that

$$\{[x]\} = \gamma, \qquad \text{if} \quad x = 2\pi \times \text{ integer } + 2\pi\gamma, \quad \gamma \in [0, 1), \tag{34}$$

we can describe the ground state by the following set of the Bethe equations

$$\lambda_j L + \sum_{k=1}^{N} \theta(\lambda_j - \lambda_k) = 2\pi \left(j - \frac{N+1}{2} \right) + 2\pi\delta, \qquad j = 1, \ldots, N, \tag{35}$$

where $\delta = \{[\pi\kappa(N-1)]\}$. Comparison of equations (35) and (29) shows that we have the following connection between the Bethe roots for periodic and twisted boundary conditions:

$$\lambda_j = \lambda_j^B + 2\pi\delta/L. \tag{36}$$

This relation is exact and holds also for the excited states if the (half)integers in the Bethe equations are the same. In the periodic case, the ground state is shifted by $2\pi\delta/L$, so that the Bethe roots are now distributed in the interval $[-q + 2\pi\delta/L, q + 2\pi\delta/L]$, and momentum of the ground state P_0 in general does not vanish,

$$P_0 = \sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} \left(\lambda_i^B + 2\pi\delta/L \right) = 2\pi D\delta. \tag{37}$$

The ground-state energy is

$$E_0 = \sum_{i=1}^{N} \lambda_i^2 = \sum_{i=1}^{N} \left(\left(\lambda_i^B\right)^2 + \frac{4\pi\delta\lambda_i^B}{L} + \frac{(2\pi\delta)^2}{L^2} \right) = E_0^B + \frac{D(2\pi\delta)^2}{L}, \tag{38}$$

where we have used that the total momentum in the case of twisted boundary conditions is zero and E_0^B in the thermodynamic limit is given by equation (32).

5. Finite-size corrections

In this section, we are going to calculate the finite-size corrections for the energy of the ground state and characteristics of the low-lying excitations. Based on the results of this section, we will be able to find the large-distance asymptotics of the correlations functions using conformal field theory. A chemical potential h is added to the Hamiltonian (1) throughout this section, so that the total Hamiltonian is

$$H_h = \int_0^L \mathrm{d}x \left\{ \left[\partial_x \Psi_A^\dagger(x)\right]\left[\partial_x \Psi_A(x)\right] + c\Psi_A^\dagger(x)\Psi_A^\dagger(x)\Psi_A(x)\Psi_A(x) - h\Psi_A^\dagger(x)\Psi_A(x) \right\}. \tag{39}$$

5.1. Finite-size corrections for the ground-state energy

As we have seen in the previous section, the ground state of the gas of anyons with twisted boundary conditions ($\beta = 1$) is characterized by the same set of Bethe equations as the Bose gas with coupling constant c' and periodic boundary conditions. So in this case we can use the results for the Bose gas [18, 24–28]:

$$E_0^B = L \int_{-q}^{q} \varepsilon_0(\lambda)\rho(\lambda)\,\mathrm{d}\lambda - \frac{\pi v_F}{6L} + \mathcal{O}\left(\frac{1}{L^2}\right), \tag{40}$$

where $\varepsilon_0(\lambda) = \lambda^2 - h$ and v_F is the Fermi velocity for the Bose gas with coupling constant c'. In the case of periodic boundary conditions ($\beta = 0$), equation (38) then gives

$$E_0 = L \int_{-q}^{q} \varepsilon_0(\lambda)\rho(\lambda)\,\mathrm{d}\lambda - \frac{\pi v_F}{6L} + \frac{D(2\pi\delta)^2}{L} + \mathcal{O}\left(\frac{1}{L^2}\right). \tag{41}$$

5.2. Low-lying excitations

In our discussion of the low-lying excitations, we consider several different types of excitation processes:

- Addition of a finite number ΔN of particles into the ground state of the system.
- Backscattering: all integers n_j in the set $\{n_j\}$ characterizing the ground-state distribution are shifted by an integer d.
- Particle–hole excitations: the integer n_j that characterizes the particle at the Fermi surface is modified from its value in the ground-state distribution by N^+ for the particle with momentum q (or $q + 2\pi\delta/L$, depending on the boundary conditions) or by N^- at the opposite point of the Fermi surface with momentum $-q$, $(-q + 2\pi\delta/L)$.

The central feature of the gas of anyons is that the boundary conditions for the field operators and the wavefunctions depend on the number of particles in the system. This means that any modification of the number of particles in the system changes the Bethe equations and, as a result, the quasiparticle momenta given by the Bethe roots. If we add one particle to the system of N particles, the boundary conditions are

$$\begin{aligned}
\chi_{N+1}(0, x_2, \ldots, x_N, x_{N+1}) &= \mathrm{e}^{\mathrm{i}\pi\beta\kappa(N-1)}\chi_{N+1}(L, x_2, \ldots, x_N, x_{N+1}),\\
\chi_{N+1}(x_1, 0, \ldots, x_N, x_{N+1}) &= \mathrm{e}^{-\mathrm{i}2\pi\kappa}\,\mathrm{e}^{\mathrm{i}\pi\beta\kappa(N-1)}\chi_{N+1}(x_1, L, \ldots, x_N, x_{N+1}),\\
&\vdots\\
\chi_{N+1}(x_1, x_2, \ldots, 0, x_{N+1}) &= \mathrm{e}^{-\mathrm{i}2N\pi\kappa}\,\mathrm{e}^{\mathrm{i}\pi\beta\kappa(N-1)}\chi_{N+1}(x_1, x_2 \ldots, L, x_{N+1}),\\
\chi_{N+1}(x_1, x_2, \ldots, x_N, 0) &= \mathrm{e}^{-\mathrm{i}2(N+1)\pi\kappa}\,\mathrm{e}^{\mathrm{i}\pi\beta\kappa(N-1)}\chi_{N+1}(x_1, x_2, \ldots, x_N, L),
\end{aligned} \tag{42}$$

and the Bethe equations become

$$\mathrm{e}^{\mathrm{i}\lambda_j L} = \mathrm{e}^{\mathrm{i}\pi\kappa N}\,\mathrm{e}^{-\mathrm{i}\pi\beta\kappa(N-1)}\prod_{k=1,k\neq j}^{N+1}\left(\frac{\lambda_j - \lambda_k + ic'}{\lambda_j - \lambda_k - ic'}\right). \tag{43}$$

The ground states for N and $(N+1)$ particles are characterized by the Bethe roots satisfying different equations

$$\begin{aligned}
\lambda_j L + \sum_{k=1}^{N}\theta(\lambda_j - \lambda_k) &= 2\pi\left(j - \frac{N+1}{2}\right) + 2\pi\omega, \qquad j = 1, \ldots, N,\\
\tilde{\lambda}_j L + \sum_{k=1}^{N+1}\theta(\tilde{\lambda}_j - \tilde{\lambda}_k) &= 2\pi\left(j - \frac{N+2}{2}\right) + 2\pi\omega', \qquad j = 1, \ldots, N+1,
\end{aligned} \tag{44}$$

where

$$\omega = 0, \quad \omega' = \kappa/2, \qquad \text{and} \qquad \omega = \{[\pi\kappa(N-1)]\}, \quad \omega' = \{[\pi\kappa N]\},$$

for the twisted ($\beta = 1$) and periodic ($\beta = 0$) boundary conditions, respectively, and $\{[\cdots]\}$ is defined by equation (34). Comparing equation (44) with equation (29) we see that

$$\lambda_j = \lambda^B_{jN} + 2\pi\omega/L, \qquad \tilde{\lambda}_j = \lambda^B_{j,N+1} + 2\pi\omega'/L, \tag{45}$$

where λ^B_{jN} are the Bethe roots characterizing the ground state of a gas of N bosons with periodic boundary conditions and coupling constant c'.

5.2.1. Addition of one particle to the system. For excitations of this type we assume that both before and after the addition of a particle, the system is in the ground state. In order to calculate the energy and momentum of this excitation, we use equation (45) which enables one to express energy and momentum through corrections to the same characteristics of excitations of the Bose gas.

For the energy we get from equation (45):

$$\Delta E(\Delta N = 1) = \sum_{j=1}^{N+1} \varepsilon_0(\tilde{\lambda}_j) - \sum_{j=1}^{N} \varepsilon_0(\lambda_j)$$
$$= \Delta E^B(\Delta N = 1) + (N+1)\left(\frac{2\pi\omega'}{L}\right)^2 - N\left(\frac{2\pi\omega}{L}\right)^2, \tag{46}$$

where ΔE^B $(\Delta N = 1)$ is the energy of the corresponding bosonic excitation. As known in the literature (see, e.g., [18, 24, 25, 27, 28]) it is convenient to express this energy in terms of the 'dressed charge' $Z(\lambda)$:

$$\Delta E^B(\Delta N = 1) = \frac{2\pi v_F}{L}\left(\frac{1}{2\mathcal{Z}}\right)^2, \tag{47}$$

where $\mathcal{Z} = Z(q) = Z(-q)$, and $Z(\lambda)$ is defined as solution of the equation

$$Z(\lambda) - \frac{1}{2\pi}\int_{-q}^{q} K(\lambda,\mu)Z(\mu)\,\mathrm{d}\mu = 1. \tag{48}$$

From (46) and (47) we obtain

$$\Delta E(\Delta N = 1) = \frac{2\pi v_F}{L}\left(\frac{1}{2\mathcal{Z}}\right)^2 + (N+1)\left(\frac{2\pi\omega'}{L}\right)^2 - N\left(\frac{2\pi\omega}{L}\right)^2. \tag{49}$$

The momentum of the excitation is

$$\Delta P(\Delta N = 1) = \sum_{j=1}^{N+1} \tilde{\lambda}_j - \sum_{j=1}^{N} \lambda_j = (N+1)\frac{2\pi\omega'}{L} - N\frac{2\pi\omega}{L}, \tag{50}$$

where we again used the fact that for the ground state of bosons with periodic boundary conditions and any number of particles the total momentum is vanishing.

5.2.2. Backscattering. The uniform shift of the ground-state distribution in a backscattering process can be understood as a jump of some number d of particles between the opposite boundaries of the Fermi surface. The Bethe equations relevant for this process (in the case of N and $(N+1)$ particles in the ground state) take the form

$$\begin{aligned} &\lambda_j^d L + \sum_{k=1}^{N} \theta\left(\lambda_j^d - \lambda_k^d\right) = 2\pi\left(j - \frac{N+1}{2}\right) + 2\pi d + 2\pi\omega, && j = 1,\dots,N,\\ &\tilde{\lambda}_j^d L + \sum_{k=1}^{N+1} \theta\left(\tilde{\lambda}_j^d - \tilde{\lambda}_k^d\right) = 2\pi\left(j - \frac{N+2}{2}\right) + 2\pi d + 2\pi\omega', && j = 1,\dots,N+1. \end{aligned} \tag{51}$$

Again, comparison with equation (29) shows that

$$\lambda_j^d = \lambda_{jN}^B + 2\pi(\omega + d)/L, \qquad \tilde{\lambda}_j^d = \lambda_{j,N+1}^B + 2\pi(\omega' + d)/L, \tag{52}$$

and the ground states are characterized by equation (45). Using equations (45) and (52) we get the excitation energy:

$$N \text{ particles:} \quad \Delta E(d) = \sum_{j=1}^{N} \left(\varepsilon_0\left(\lambda_j^d\right) - \varepsilon_0(\lambda_j)\right) = N\frac{(2\pi\omega + 2\pi d)^2}{L^2} - N\frac{(2\pi\omega)^2}{L^2}, \tag{53}$$

$$\begin{aligned} N+1 \text{ particles:} \quad \Delta E(d) &= \sum_{j=1}^{N+1} \left(\varepsilon_0\left(\tilde{\lambda}_j^d\right) - \varepsilon_0(\tilde{\lambda}_j)\right) \\ &= (N+1)\frac{(2\pi\omega' + 2\pi d)^2}{L^2} - (N+1)\frac{(2\pi\omega')^2}{L^2}. \end{aligned} \tag{54}$$

This result can be rewritten using the relation $\mathcal{Z}^2 = 2\pi D/v_F$ (see [18, chapter I.9]) obtaining

$$\begin{aligned} &N \text{ particles:} \quad \Delta E(d) = \frac{2\pi v_F}{L}\mathcal{Z}^2(d+\omega)^2 - \frac{2\pi v_F}{L}\mathcal{Z}^2\omega^2, \\ &N+1 \text{ particles:} \quad \Delta E(d) = \frac{2\pi v_F}{L}\mathcal{Z}^2(d+\omega')^2 - \frac{2\pi v_F}{L}\mathcal{Z}^2\omega'^2 + \frac{(2\pi\omega'+2\pi d)^2}{L^2} - \frac{(2\pi\omega')^2}{L^2}. \end{aligned} \tag{55}$$

The momentum of the backscattering excitation is simply

$$\Delta P(d) = N(2\pi d/L), \tag{56}$$

the expression that is valid for any number of particles N.

5.2.3. Particle–hole excitations at the Fermi surface. In this case, the excitations we consider consist in changing the maximal (minimal) n_j in the ground state by $N^{\pm}$. For N particles and 'excitation magnitude' N^+ the Bethe equations are

$$\begin{aligned} &\lambda_j L + \sum_{k=1}^{N} \theta(\lambda_j - \lambda_k) = 2\pi\left(j - \frac{N+1}{2}\right) + 2\pi\omega, \qquad j = 1, \ldots, N-1, \\ &\lambda_N L + \sum_{k=1}^{N} \theta(\lambda_N - \lambda_k) = 2\pi\left(N - \frac{N+1}{2}\right) + 2\pi\omega + 2\pi N^+. \end{aligned} \tag{57}$$

From (57) we see that the momentum of the excitation N^+ is $\Delta P(N^+) = 2\pi N^+/L$ and, similarly, for the excitation N^- the momentum is $\Delta P(N^-) = -2\pi N^-/L$. These excitation can be considered as a special case of the general particle–hole excitations, and we can use the results of appendix B for them. Using (B.7) we see that the excitation energy and momentum

$$\Delta E(N^{\pm}) = \frac{2\pi v_F}{L}N^{\pm} + \mathcal{O}\left(\frac{1}{L^2}\right), \qquad \Delta P(N^{\pm}) = \pm\frac{2\pi}{L}N^{\pm}, \tag{58}$$

coincide with those for the similar excitations of the Bose gas (see appendix I.4 of [18]):

$$\Delta E^B(N^{\pm}) = \frac{2\pi v_F}{L}N^{\pm}, \qquad \Delta P^B(N^{\pm}) = \pm\frac{2\pi}{L}N^{\pm}. \tag{59}$$

For $(N+1)$ particles, the energy and momentum of the excitations are given by the same expressions as in (58).

6. Large-distance asymptotics of correlation functions

In this section, we calculate the asymptotics of the correlation functions. We will consider the case of twisted boundary conditions ($\beta = 1$), or the periodic boundary conditions ($\beta = 0$) when κ is a integer multiple of $2/(N-1)$, so that the shift in (36) vanishes, $\delta = 0$, and the two boundary conditions are equivalent (see (10)). The main feature of this case that is important for the direct applicability of the conformal field theory approach is that the momentum of the ground state (37) of the gas of anyons is zero for these boundary conditions. For general gapless (1+1)-dimensional systems, $T = 0$ is a critical point making the correlation functions decay as a power of distance at $T = 0$ but exponentially at $T > 0$. As we have seen in the previous section, the Lieb-Liniger anyonic gas is gapless and the excitation spectrum has a linear dispersion law in the vicinity of the Fermi level. These features support the expectation that the critical behavior of the anyon system is described by conformal field theory (CFT).

CFT is a vast subject and we refer the reader to [29–32] and [18, chapter XVIII] for more information. A conformal theory is characterized by the central charge c (not to be confused with the coupling constant in (1)) of the underlying Virasoro algebra, and conformal invariance constrains the critical behavior of the systems under consideration. The critical exponents (the powers that characterizes the algebraic decay at $T = 0$) are related to the conformal dimensions of the operators within the CFT, so to obtain the complete information about the critical behavior of the system we need to calculate the central charge and the conformal dimensions of the primary fields.

6.1. Central charge

In order to find the central charge we use the fact that for unitary conformal theories it can be found from the finite-size corrections, specifically the coefficient of the $1/L$ term in the expansion of the ground-state energy for $L \to \infty$ [33, 34]:

$$E = L\epsilon_\infty - \frac{\pi v_F}{6L}c + \mathcal{O}\left(\frac{1}{L}\right). \tag{60}$$

Comparing this relation to equation (40) valid for the boundary conditions we are assuming in this section, we see that the central charge $c = 1$. The fact that the central charge $c = 1$ means that the critical exponents can depend continuously on the parameters of the model [29, 35, 36].

6.2. Conformal dimensions from finite-size effects

Following the original idea of Cardy [37] subsequently developed in [24, 25, 27], we obtain below the conformal dimensions of the conformal fields in the theory from the spectrum of the low-lying excitations described in the previous section. The local fields of the model can be represented as a combination of conformal fields

$$\phi(x,t) = \sum_Q \tilde{\tilde{A}}(Q)\phi_Q(z,\bar{z}), \tag{61}$$

where $\tilde{\tilde{A}}(Q)$ are some coefficients and $z = \mathrm{i}x + v_F\tau$, with v_F the Fermi velocity and τ the Euclidean time. The conformal fields are related to excitations with quantum numbers $Q = \{\Delta N, N^\pm, d\}$, where ΔN represents the number of particles created by the field ϕ, and all the fields in the expansion (61) should have the same ΔN. The quantum number d gives the number of particles backscattered across the Fermi 'sphere', and $N^\pm$ characterizes the change of the maximal or minimal n_j in the Bethe equations from its values in the ground state. While ΔN has to be the same for all the terms in the expansion, d and $N^\pm$ can be different.

For two conformal fields, ϕ_Q and $\phi_{Q'}$, with the same conformal dimensions denoted $\Delta^\pm$, their correlation function is given by

$$\langle\phi_Q(z_1,\bar{z}_1)\phi_{Q'}(z_2,\bar{z}_2)\rangle = \frac{1}{(z_1-z_2)^{2\Delta^+}(\bar{z}_1-\bar{z}_2)^{2\Delta^-}}. \tag{62}$$

Under a conformal transformation $z = z(w), \bar{z} = \bar{z}(\bar{w})$, it transforms like

$$\langle \phi_Q(w_1, \bar{w}_1)\phi_{Q'}(w_2, \bar{w}_2)\rangle = \prod_{j=1}^{2} \left(\frac{\partial z_j}{\partial w_j}\right)^{\Delta^+} \left(\frac{\partial \bar{z}_j}{\partial \bar{w}_j}\right)^{\Delta^-} \times \langle \phi_Q(z_1(w_1), \bar{z}_1(\bar{w}_1))\phi_{Q'}(z_2(w_2), \bar{z}_2(\bar{w}_2))\rangle. \tag{63}$$

Using the expansion (61), the fact that the two conformal fields with different conformal dimensions are orthogonal (their correlation function is zero), and (62) we then have

$$\langle \phi(z_1, \bar{z}_1)\phi(z_2, \bar{z}_2)\rangle = \sum_Q \frac{\tilde{A}(Q)}{(z_1 - z_2)^{2\Delta_Q^+}(\bar{z}_1 - \bar{z}_2)^{2\Delta_Q^-}}, \tag{64}$$

which is valid in the whole complex plane without the origin ($z_1 \neq z_2$). Conformal mapping of this plane to a cylinder (periodic strip) with the help of transformation

$$z = \mathrm{e}^{2\pi w/L}, \qquad w = \mathrm{i}x + v_F\tau \qquad \text{with} \quad 0 < x \leqslant L, \tag{65}$$

applied to (63) gives

$$\langle \phi(w_1, \bar{w}_1)\phi(w_2, \bar{w}_2)\rangle = \sum_Q \tilde{A}(Q) \left(\frac{\pi/L}{\sinh[\pi(w_1 - w_2)/L]}\right)^{2\Delta_Q^+} \times \left(\frac{\pi/L}{\sinh[\pi(\bar{w}_1 - \bar{w}_2)/L]}\right)^{2\Delta_Q^-}, \tag{66}$$

with the asymptotics

$$\langle \phi(w_1, \bar{w}_1)\phi(w_2, \bar{w}_2)\rangle \sim \sum_Q \mathrm{e}^{-\frac{2\pi v_F}{L}(\Delta_Q^+ + \Delta_Q^-)(\tau_1 - \tau_2) - \mathrm{i}\frac{2\pi}{L}(\Delta_Q^+ - \Delta_Q^-)(x_1 - x_2)}. \tag{67}$$

Comparison with the spectral decomposition of the correlation function in the periodic strip ($\tau_1 > \tau_2$)

$$\langle \phi(w_1, \bar{w}_1)\phi(w_2, \bar{w}_2)\rangle_L = \sum_Q |\langle 0|\phi(0,0)|Q\rangle|^2 \,\mathrm{e}^{-(E_Q - E_0)(\tau_1 - \tau_2) - i(P_Q - P_0)(x_1 - x_2)}, \tag{68}$$

where $|0\rangle$ is the ground state and E_0, P_0 are the energy and momentum of the ground state, respectively, leads to

$$E_Q - E_0 = \frac{2\pi v_F}{L}\left(\Delta_Q^+ + \Delta_Q^-\right), \qquad P_Q - P_0 = \frac{2\pi}{L}\left(\Delta_Q^+ - \Delta_Q^-\right), \tag{69}$$

assuming that both the energy and momentum gaps are of order $\mathcal{O}(1/L)$. However, as we have seen in Sect. 5, for some of the excitations considered (addition of a particle in the system, $\Delta N = 1$, backscattering processes characterized by d, and particle–hole excitations at the Fermi surface characterized by $N^{\pm}$), the momentum gap is macroscopic. For example, if $Q = \{\Delta N = 0, d \neq 0, N^{\pm} = 0\}$, the momentum gap is $2k_F d, k_F \equiv \pi D$, and for $Q = \{\Delta N = 1, d = 0, N^{\pm} = 0\}$ the momentum gap is $\pi k_F \kappa + \pi\kappa/L$. For these excitations, following [24, 25, 27], the coefficients $\tilde{A}(Q)$ will depend on x as

$$\tilde{A}(Q) = A(Q)\,\mathrm{e}^{\mathrm{i}p_Q x}, \tag{70}$$

where p_Q is the macroscopic part of the momentum gap $P_Q - P_0$. From (64) and (70) we obtain the generic formula for the asymptotics of correlations functions at $T = 0$

$$\langle \phi(x,t)\phi(0,0)\rangle = \sum_Q \frac{A(Q)\,\mathrm{e}^{\mathrm{i}p_Q x}}{(\mathrm{i}x + v_F\tau)^{2\Delta_Q^+}(-\mathrm{i}x + v_F\tau)^{2\Delta_Q^-}}, \tag{71}$$

where $\Delta_Q^{\pm}$ can be found from (69) and the leading term corresponds to the smallest $\Delta_Q^{\pm}$.

We also can find the low-temperature asymptotics of the correlation functions if we use instead of the conformal mapping (65), the mapping

$$z = \mathrm{e}^{2\pi T w/v_F}, \qquad z = x - i v_F \tau, \tag{72}$$

which differ from (65) by interchanging the space and time variables. The computations are similar to those described above for the correlation functions in a finite box, and the final result is

$$\langle\phi(x,t)\phi(0,0)\rangle_T = \sum_Q B(Q)\,\mathrm{e}^{\mathrm{i}p_Q x}\left(\frac{\pi T/v_F}{\sinh[\pi T(x - i v_F\tau)/v_F]}\right)^{2\Delta_Q^+} \times \left(\frac{\pi T/v_F}{\sinh[\pi T(x + i v_F\tau)/v_F]}\right)^{2\Delta_Q^-}. \tag{73}$$

This result is valid only at temperatures close to zero.

6.3. Density correlation function

In the case of the density correlation function, $\langle j(x,t)j(0,0)\rangle$, where $j(x) = \Psi_A^\dagger(x)\Psi_A(x)$, we have $\Delta N = 0$ so the most general excitation is constructed by backscattering d particles and creating a particle–hole pair at the Fermi surface characterized by $N^\pm$. Making use of (55), (56), (58), we obtain for the energy and momentum gap of the excitation characterized by $Q = \{\Delta N = 0, d, N^\pm\}$:

$$P_{N^\pm,d} - P_0 = 2k_F d + \frac{2\pi}{L}(N^+ - N^-), \tag{74}$$

$$E_{N^\pm,d} - E_0 = \frac{2\pi v_F}{L}[(\mathcal{Z}d)^2 + N^+ + N^-]. \tag{75}$$

Here we have taken into account only the terms of order 1 and $\mathcal{O}(1/L)$. Equation (69) gives the conformal dimensions

$$2\Delta_Q^\pm = 2N^\pm + (\mathcal{Z}d)^2, \tag{76}$$

and from the general formula (71)

$$\langle j(x,t)j(0,0)\rangle - \langle j(0,0)\rangle^2 = \sum_{Q=\{N^\pm,d\}} A(Q)\frac{\mathrm{e}^{2\mathrm{i}xk_F d}}{(\mathrm{i}x + v_F\tau)^{2\Delta_Q^+}(-\mathrm{i}x + v_F\tau)^{2\Delta_Q^-}}. \tag{77}$$

Defining $\theta \equiv 2\mathcal{Z}^2 = 4\pi D/v_F$, where $\mathcal{Z} = Z(-q) = Z(q)$, and $Z(\lambda)$ given by the integral equation (48), the leading terms are

$$\langle j(x,t)j(0,0)\rangle - \langle j(0,0)\rangle^2 = \frac{a}{(\mathrm{i}x + v_F\tau)^2} + \frac{a}{(-\mathrm{i}x + v_F\tau)^2} + b\frac{\cos(2k_F x)}{|\mathrm{i}x + v_F\tau|^\theta}. \tag{78}$$

For equal times, equation (77) takes the form

$$\langle j(x,0)j(0,0)\rangle - \langle j(0,0)\rangle^2 = \sum_{Q=\{N^\pm,d\}} \hat{A}(Q)\frac{\mathrm{e}^{2\mathrm{i}xk_F d}}{|x|^{d^2\theta+2N^+ + 2N^-}}. \tag{79}$$

The presence of the oscillatory terms in this expression can be explained by the following simple computation [27]:

$$\langle j(x,0)j(0,0)\rangle = \sum_Q \langle 0|j(x,0)|Q\rangle\langle Q|j(0,0)|0\rangle = \sum_Q |\langle 0|j(0,0)|Q\rangle|^2\,\mathrm{e}^{\mathrm{i}(P_Q - P_0)x}$$
$$= \sum_{d=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}2k_F\,\mathrm{d}x}\sum_{N^\pm} |\langle 0|j(0,0)|d, N^\pm\rangle|^2\,\mathrm{e}^{\frac{\mathrm{i}2\pi x}{L}(N^+ - N^-)}, \tag{80}$$

where in the second line, we broke the sum over Q into disjoint sums characterized by different macroscopic momenta. The second part of the sum gives the power-law decay for $k_F^{-1} \ll x \ll L$. The formulae (77) and (79) are the same as in the case of a Bose gas with coupling constant $c' = c/\cos(\pi\kappa/2)$ and periodic boundary conditions [27] (see chapter XVII of [18]). This situation is expected, since

$$j(x) = \Psi_A^\dagger(x)\Psi_A(x) = \Psi_B^\dagger(x)\Psi_B(x). \tag{81}$$

Finally, from (73), the finite temperature density correlation function is

$$\langle j(x,t)j(0,0)\rangle_T = \sum_{Q=\{d,N^\pm\}} B(Q)\,\mathrm{e}^{\mathrm{i}2k_F\,\mathrm{d}x}\left(\frac{\pi T/v_F}{\sinh[\pi T(x-\mathrm{i}v_F\tau)/v_F]}\right)^{2\Delta_Q^+} \times\left(\frac{\pi T/v_F}{\sinh[\pi T(x+\mathrm{i}v_F\tau)/v_F]}\right)^{2\Delta_Q^-}, \tag{82}$$

with $\Delta_Q^\pm$ given by (76).

6.4. Field-field correlator

In contrast to the density correlators, for the field correlator $\langle\Psi_A(x,t)\Psi_A^\dagger(0,0)\rangle$, one has $\Delta N = 1$. For the ground states with N and $(N+1)$ particles and the boundary conditions considered in this section the Bethe equations are

$$\begin{aligned}\lambda_j L + \sum_{k=1}^{N}\theta(\lambda_j-\lambda_k) &= 2\pi\left(j-\frac{N+1}{2}\right), \qquad j=1,\ldots,N,\\ \tilde\lambda_j L + \sum_{k=1}^{N+1}\theta(\tilde\lambda_j-\tilde\lambda_k) &= 2\pi\left(j-\frac{N+2}{2}\right)+\pi\kappa, \qquad j=1,\ldots,N+1.\end{aligned} \tag{83}$$

The shift $\pi\kappa$ in the second equation implies that the anyonic wavefunctions for N and $(N+1)$ particles live in two orthogonal sectors of the Hilbert space. The addition of one particle produces in this case a macroscopic change in the momentum, $\pi k_F\kappa + \pi\kappa/L$, which gives rise to oscillations even in the dominant term of the field correlator.

The most general excitation is obtained by an addition of one particle to the system, followed by the backscattering of d particles and creation of a particle–hole pair at the Fermi surface. Using the results (49), (50), (55), (56), (58) with $\omega = 0, \omega' = \kappa/2$, we obtain the following expressions for the energy and momentum gaps of an excitation with $Q = \{\Delta N = 1, d, N^\pm\}$ (retaining, as before, the terms of order 1 and $\mathcal{O}(1/L)$):

$$P_{N^\pm,d}^{\Delta N=1} - P_0 = 2k_F(d+\kappa/2) + \frac{2\pi}{L}[(d+\kappa/2)+N^+-N^-], \tag{84}$$

$$E_{N^\pm,d}^{\Delta N=1} - E_0 = \frac{2\pi v_F}{L}\left[\left(\frac{1}{2\mathcal{Z}}\right)^2 + \mathcal{Z}^2(d+\kappa/2)^2 + N^+ + N^-\right], \tag{85}$$

so the conformal dimensions are

$$2\Delta_Q^\pm = 2N^\pm + \left(\frac{1}{2\mathcal{Z}} \pm \mathcal{Z}(d+\kappa/2)\right)^2. \tag{86}$$

From equation (71), the field correlator is

$$\left\langle\Psi_A(x,t)\Psi_A^\dagger(0,0)\right\rangle = \sum_{Q=\{N^\pm,d\}} A(Q)\frac{\mathrm{e}^{2\mathrm{i}k_F(d+\frac{\kappa}{2})x}}{(\mathrm{i}x+v_F\tau)^{2\Delta_Q^+}(-\mathrm{i}x+v_F\tau)^{2\Delta_Q^-}}, \tag{87}$$

or in the equal-time case

$$\left\langle \Psi_A(x,0)\Psi_A^{\dagger}(0,0)\right\rangle = \sum_{Q=\{N^{\pm},d\}} \hat{A}(Q)\frac{\mathrm{e}^{2\mathrm{i}k_F(d+\frac{\kappa}{2})x}}{|x|^{(d+\frac{\kappa}{2})^2\theta+\frac{1}{\theta}+2N^{+}+2N^{-}}}, \tag{88}$$

where $\theta = 2\mathcal{Z}^2$. Again, we can heuristically justify the presence of the oscillatory terms in the correlation function in the same way as for the density correlator, but for the field correlator, the complete set of states that is inserted between Ψ_A and $\Psi_A^{\dagger}$ is from the sector with $(N+1)$ particles

$$\begin{aligned}\left\langle \Psi_A(x,0)\Psi_A^{\dagger}(0,0)\right\rangle &= \sum_Q \langle 0|\Psi_A(x,0)|Q\rangle\langle Q|\Psi_A^{\dagger}(0,0)|0\rangle \\ &= \sum_Q |\langle 0|\Psi_A(0,0)|Q\rangle|^2\, \mathrm{e}^{\mathrm{i}(P_Q-P_0)x} \\ &= \sum_{d=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}2k_F(d+\frac{\kappa}{2})x} \sum_{N^{\pm}} |\langle 0|\Psi_A(0,0)|d,N^{\pm}\rangle|^2\, \mathrm{e}^{\frac{\mathrm{i}2\pi x}{L}(N^{+}-N^{-})}.\end{aligned} \tag{89}$$

In this case, the terms of the correlation function containing $\mathrm{e}^{\mathrm{i}2k_F(d+\kappa/2)x}$ that are responsible for the oscillatory behavior at $x \ll L$, exhibit dependence on the statistical parameter.

Equation (87) can be compared to the result of Calabrese and Mintchev [19], who calculated the field correlation function for anyonic gapless systems in the low-momentum regime using the harmonic fluid approach [38, 39], obtaining

$$\left\langle \Psi_A^{\dagger}(x,0)\Psi_A(0,0)\right\rangle = D\sum_{d=-\infty}^{\infty} b_d \frac{\mathrm{e}^{-2\mathrm{i}(d+\frac{\kappa}{2})k_F x}\,\mathrm{e}^{-2\mathrm{i}(m+\frac{\kappa}{2})\pi\epsilon(x)/2}}{(Dc(x))^{(d+\frac{\kappa}{2})^2 2K+\frac{1}{2K}}}, \tag{90}$$

where D is the density, b_d unknown non-universal amplitudes, $c(x) = L\sin(\pi x/L)$ and K is a universal parameter that can be expressed in terms of the phenomenological velocity parameters v_N, v_J as $K = \sqrt{v_J/v_N}$. For the Lieb–Liniger anyons,

$$K = \frac{2\pi D}{v_F} = \frac{\theta}{2}. \tag{91}$$

They have checked their results in the limit $c \to \infty$, $K = 1$ against the exact results of Santachiara *et al* [40], who calculated the generalization of Lenard formula [41] for anyonic statistics. We see that our conformal field theory approach agrees with the leading asymtotics produced by the harmonic liquid approximation but also gives the higher-order terms in the large-distance expansion.

Using the conformal mapping (72) that leads to general equation (73), we find also the finite-temperature field correlator:

$$\begin{aligned}\langle \Psi_A(x,t)\Psi_A^{\dagger}(0,0)\rangle_T = &\sum_{Q=\{d,N^{\pm}\}} B(Q)\,\mathrm{e}^{\mathrm{i}2k_F(d+\frac{\kappa}{2})x}\left(\frac{\pi T/v_F}{\sinh[\pi T(x-\mathrm{i}v_F\tau)/v_F]}\right)^{2\Delta_Q^{+}} \\ &\times\left(\frac{\pi T/v_F}{\sinh[\pi T(x+\mathrm{i}v_F\tau)/v_F]}\right)^{2\Delta_Q^{-}},\end{aligned} \tag{92}$$

where $\Delta_Q^{\pm}$ is given by (86).

7. Conclusions

The main result of our work is the calculation of the large-distance asymptotics of the correlation functions of the gas of 1D anyons using the ideas of conformal field theory.

This result requires conformal invariance close to the critical point $T = 0$, and the knowledge of the finite-size corrections to the energy and momentum of the ground state of the gas due to low-lying excitations. In the analogous case of Bose gas with δ-function repulsive interaction, the conformal field theory predictions for the asymptotics of the correlators were checked against the exact results for these asymptotics obtained from the determinant representations and the differential equations for the correlation functions [18]. It would be interesting to have similar exact results for the model studied in this paper which is a natural anyonic extension of the Bose gas. As a first step in this direction, Santachiara, Stauffer and Cabra [40], already obtained for the one-particle reduced density matrix (field correlator) in the impenetrable limit a representation in terms of the determinant of a Toeplitz matrix of dimension $(N-1)\times(N-1)$ where N is the number of particles. The exact results for the anyon correlation functions would also be needed to extend the correlators derived in this work for essentially one type of boundary conditions to more general quasiperiodic conditions. This problem seems particularly natural for anyons for which the effective boundary conditions for quasiparticle momenta change with the total number of particles in the system.

Acknowledgments

This work was supported in part by the NSF grants DMR-0325551, DMR-0302758 and DMS-0503712.

Appendix A. Boundary conditions for the multi-anyon wavefunctions

In this appendix, we derive the exact form of the cyclic boundary conditions for the wavefunctions of the many-anyon system. Our treatment generalizes the approach of [6] to the case of several penetrable particles. In physical terms, the situation we consider corresponds to anyons confined to move along a loop with, in general, an external phase shift ϕ created, e.g., by a magnetic field threading the loop. We start with the case of *two particles* and no external phase shift, $\phi = 0$. The Bethe-Anzatz wavefunction (24) reduces in this case to the following form: in the region I ($x_1 < x_2$) one has

$$\chi_{\mathrm{I}}(x_1, x_2) = \frac{\mathrm{e}^{\mathrm{i}\pi\kappa/2}}{\sqrt{2[(\lambda_2-\lambda_1)^2 + c'^2]}}\{\mathrm{e}^{\mathrm{i}(x_1\lambda_1+x_2\lambda_2)}(\lambda_2-\lambda_1-\mathrm{i}c') + \mathrm{e}^{\mathrm{i}(x_1\lambda_2+x_2\lambda_1)}(\lambda_2-\lambda_1+\mathrm{i}c')\}, \tag{A.1}$$

and in the region II ($x_1 > x_2$):

$$\chi_{\mathrm{II}}(x_1, x_2) = \frac{\mathrm{e}^{-\mathrm{i}\pi\kappa/2}}{\sqrt{2[(\lambda_2-\lambda_1)^2 + c'^2]}}\{\mathrm{e}^{\mathrm{i}(x_1\lambda_1+x_2\lambda_2)}(\lambda_2-\lambda_1+\mathrm{i}c') + \mathrm{e}^{\mathrm{i}(x_1\lambda_2+x_2\lambda_1)}(\lambda_2-\lambda_1-\mathrm{i}c')\}. \tag{A.2}$$

The general exchange symmetry of this wavefunction given by equation (17) imply that for fractional κ it cannot satisfy the same boundary conditions in the two coordinates. As one can see by exchanging the coordinates, if the wavefunction is periodic in the first one, the boundary conditions in second one should have a twist,

$$\chi(0, x) = \chi(L, x) \rightarrow \chi(x, 0) = \chi(x, L)\,\mathrm{e}^{-2\mathrm{i}\pi\kappa} \tag{A.3}$$

and viceversa. One consequence of this is that the exact form of the Bethe equations (26) depends on whether we impose periodic boundary conditions on one or the other coordinate. Indeed, if one requires periodicity in x_1, $\chi(0, x_2) = \chi(L, x_2)$, the Bethe equations are

$$e^{iL\lambda_j} = e^{i\pi\kappa} \prod_{k=1, k\neq j}^{2} \left(\frac{\lambda_j - \lambda_k + ic'}{\lambda_j - \lambda_k - ic'} \right), \tag{A.4}$$

whereas the periodicity in x_2, $\chi(x_1, 0) = \chi(x_1, L)$, results in the equations that differ by the sign of the statistics parameter κ,

$$e^{iL\lambda_j} = e^{-i\pi\kappa} \prod_{k=1, k\neq j}^{2} \left(\frac{\lambda_j - \lambda_k + ic'}{\lambda_j - \lambda_k - ic'} \right). \tag{A.5}$$

Since the Bethe equations determine the spectrum of the quasiparticle momenta λ_j through equation (27), the κ shifts of different signs produce two physically different situations.

The origin of this difference can be traced back to the fact that the fractional statistics requires braiding of particles, something that strictly speaking cannot be done in one dimension. To define the braiding of 1D particles one needs to first adopt a convention on how the particles pass each other at coinciding points, something that is done by choosing a specific sign of the exchange phase $e^{-i\pi\kappa\epsilon(x_1-x_2)/2}$. After that, one more choice that needs to be made is how the 1D loop with anyons is imbedded into the underlying 2D anyonic system. In the case of two particles, this choice is reflected in the possibility of choosing different boundary conditions for two different anyonic coordinates and determines how the particle trajectories enclose each other as the particles move along the loop [6]. As reflected in equation (A.3), periodicity in x_1 means that the trajectory of x_1 does not enclose the particle x_2. This implies that x_1 is itself enclosed by the trajectory of x_2, producing the twist in the boundary condition for x_2 variable. The different choice of the boundary condition would mean that the 1D loop in imbedded into the 2D system in such a way that the trajectory of x_1 encloses x_2. This means that the wavefunction periodicity in both variables correspond to different but valid physical situations.

The situation is somewhat more complicated for larger number of particles, as can be seen in the case of *three particles*. In the wavefunction (24), one needs to distinguish then six regions corresponding to the six permutation of the particles. The wavefunction (24) in these regions is

Region I ($x_1 < x_2 < x_3$)

$$\begin{aligned}
\chi_{\mathrm{I}}(x_1, x_2, x_3) = A\, e^{\frac{i3\pi\kappa}{2}} \{ & e^{i(x_1\lambda_1+x_2\lambda_2+x_3\lambda_3)}(\lambda_3-\lambda_2-ic')(\lambda_3-\lambda_1-ic')(\lambda_2-\lambda_1-ic') \\
& - e^{i(x_1\lambda_1+x_2\lambda_3+x_3\lambda_2)}(\lambda_2-\lambda_3-ic')(\lambda_2-\lambda_1-ic')(\lambda_3-\lambda_1-ic') \\
& + e^{i(x_1\lambda_3+x_2\lambda_1+x_3\lambda_2)}(\lambda_2-\lambda_1-ic')(\lambda_2-\lambda_3-ic')(\lambda_1-\lambda_3-ic') \\
& - e^{i(x_1\lambda_3+x_2\lambda_2+x_3\lambda_1)}(\lambda_1-\lambda_2-ic')(\lambda_1-\lambda_3-ic')(\lambda_2-\lambda_3-ic') \\
& + e^{i(x_1\lambda_2+x_2\lambda_3+x_3\lambda_1)}(\lambda_1-\lambda_3-ic')(\lambda_1-\lambda_2-ic')(\lambda_3-\lambda_2-ic') \\
& - e^{i(x_1\lambda_2+x_2\lambda_1+x_3\lambda_3)}(\lambda_3-\lambda_1-ic')(\lambda_3-\lambda_2-ic')(\lambda_1-\lambda_2-ic') \},
\end{aligned} \tag{A.6}$$

Region II ($x_1 < x_3 < x_2$)

$$\begin{aligned}
\chi_{\mathrm{II}}(x_1, x_2, x_3) = A\, e^{\frac{i\pi\kappa}{2}} \{ & e^{i(x_1\lambda_1+x_2\lambda_2+x_3\lambda_3)}(\lambda_3-\lambda_2+ic')(\lambda_3-\lambda_1-ic')(\lambda_2-\lambda_1-ic') \\
& - e^{i(x_1\lambda_1+x_2\lambda_3+x_3\lambda_2)}(\lambda_2-\lambda_3+ic')(\lambda_2-\lambda_1-ic')(\lambda_3-\lambda_1-ic') \\
& + e^{i(x_1\lambda_3+x_2\lambda_1+x_3\lambda_2)}(\lambda_2-\lambda_1+ic')(\lambda_2-\lambda_3-ic')(\lambda_1-\lambda_3-ic') \\
& - e^{i(x_1\lambda_3+x_2\lambda_2+x_3\lambda_1)}(\lambda_1-\lambda_2+ic')(\lambda_1-\lambda_3-ic')(\lambda_2-\lambda_3-ic') \\
& + e^{i(x_1\lambda_2+x_2\lambda_3+x_3\lambda_1)}(\lambda_1-\lambda_3+ic')(\lambda_1-\lambda_2-ic')(\lambda_3-\lambda_2-ic') \\
& - e^{i(x_1\lambda_2+x_2\lambda_1+x_3\lambda_3)}(\lambda_3-\lambda_1+ic')(\lambda_3-\lambda_2-ic')(\lambda_1-\lambda_2-ic') \},
\end{aligned} \tag{A.7}$$

Region III ($x_3 < x_1 < x_2$)

$$\chi_{\rm III}(x_1,x_2,x_3) = A\,{\rm e}^{\frac{-{\rm i}\pi\kappa}{2}}\{{\rm e}^{{\rm i}(x_1\lambda_1+x_2\lambda_2+x_3\lambda_3)}(\lambda_3-\lambda_2+{\rm i}c')(\lambda_3-\lambda_1+{\rm i}c')(\lambda_2-\lambda_1-{\rm i}c')$$
$$-\,{\rm e}^{{\rm i}(x_1\lambda_1+x_2\lambda_3+x_3\lambda_2)}(\lambda_2-\lambda_3+{\rm i}c')(\lambda_2-\lambda_1+{\rm i}c')(\lambda_3-\lambda_1-{\rm i}c')$$
$$+\,{\rm e}^{{\rm i}(x_1\lambda_3+x_2\lambda_1+x_3\lambda_2)}(\lambda_2-\lambda_1+{\rm i}c')(\lambda_2-\lambda_3+{\rm i}c')(\lambda_1-\lambda_3-{\rm i}c')$$
$$-\,{\rm e}^{{\rm i}(x_1\lambda_3+x_2\lambda_2+x_3\lambda_1)}(\lambda_1-\lambda_2+{\rm i}c')(\lambda_1-\lambda_3+{\rm i}c')(\lambda_2-\lambda_3-{\rm i}c')$$
$$+\,{\rm e}^{{\rm i}(x_1\lambda_2+x_2\lambda_3+x_3\lambda_1)}(\lambda_1-\lambda_3+{\rm i}c')(\lambda_1-\lambda_2+{\rm i}c')(\lambda_3-\lambda_2-{\rm i}c')$$
$$-\,{\rm e}^{{\rm i}(x_1\lambda_2+x_2\lambda_1+x_3\lambda_3)}(\lambda_3-\lambda_1+{\rm i}c')(\lambda_3-\lambda_2+{\rm i}c')(\lambda_1-\lambda_2-{\rm i}c')\}, \quad \text{(A.8)}$$

Region IV ($x_3 < x_2 < x_1$)

$$\chi_{\rm IV}(x_1,x_2,x_3) = A\,{\rm e}^{\frac{-{\rm i}3\pi\kappa}{2}}\{{\rm e}^{{\rm i}(x_1\lambda_1+x_2\lambda_2+x_3\lambda_3)}(\lambda_3-\lambda_2+{\rm i}c')(\lambda_3-\lambda_1+{\rm i}c')(\lambda_2-\lambda_1+{\rm i}c')$$
$$-\,{\rm e}^{{\rm i}(x_1\lambda_1+x_2\lambda_3+x_3\lambda_2)}(\lambda_2-\lambda_3+{\rm i}c')(\lambda_2-\lambda_1+{\rm i}c')(\lambda_3-\lambda_1+{\rm i}c')$$
$$+\,{\rm e}^{{\rm i}(x_1\lambda_3+x_2\lambda_1+x_3\lambda_2)}(\lambda_2-\lambda_1+{\rm i}c')(\lambda_2-\lambda_3+{\rm i}c')(\lambda_1-\lambda_3+{\rm i}c')$$
$$-\,{\rm e}^{{\rm i}(x_1\lambda_3+x_2\lambda_2+x_3\lambda_1)}(\lambda_1-\lambda_2+{\rm i}c')(\lambda_1-\lambda_3+{\rm i}c')(\lambda_2-\lambda_3+{\rm i}c')$$
$$+\,{\rm e}^{{\rm i}(x_1\lambda_2+x_2\lambda_3+x_3\lambda_1)}(\lambda_1-\lambda_3+{\rm i}c')(\lambda_1-\lambda_2+{\rm i}c')(\lambda_3-\lambda_2+{\rm i}c')$$
$$-\,{\rm e}^{{\rm i}(x_1\lambda_2+x_2\lambda_1+x_3\lambda_3)}(\lambda_3-\lambda_1+{\rm i}c')(\lambda_3-\lambda_2+{\rm i}c')(\lambda_1-\lambda_2+{\rm i}c')\}, \quad \text{(A.9)}$$

Region V ($x_2 < x_1 < x_3$)

$$\chi_{\rm V}(x_1,x_2,x_3) = A\,{\rm e}^{\frac{{\rm i}\pi\kappa}{2}}\{{\rm e}^{{\rm i}(x_1\lambda_1+x_2\lambda_2+x_3\lambda_3)}(\lambda_3-\lambda_2-{\rm i}c')(\lambda_3-\lambda_1-{\rm i}c')(\lambda_2-\lambda_1+{\rm i}c')$$
$$-\,{\rm e}^{{\rm i}(x_1\lambda_1+x_2\lambda_3+x_3\lambda_2)}(\lambda_2-\lambda_3-{\rm i}c')(\lambda_2-\lambda_1-{\rm i}c')(\lambda_3-\lambda_1+{\rm i}c')$$
$$+\,{\rm e}^{{\rm i}(x_1\lambda_3+x_2\lambda_1+x_3\lambda_2)}(\lambda_2-\lambda_1-{\rm i}c')(\lambda_2-\lambda_3-{\rm i}c')(\lambda_1-\lambda_3+{\rm i}c')$$
$$-\,{\rm e}^{{\rm i}(x_1\lambda_3+x_2\lambda_2+x_3\lambda_1)}(\lambda_1-\lambda_2-{\rm i}c')(\lambda_1-\lambda_3-{\rm i}c')(\lambda_2-\lambda_3+{\rm i}c')$$
$$+\,{\rm e}^{{\rm i}(x_1\lambda_2+x_2\lambda_3+x_3\lambda_1)}(\lambda_1-\lambda_3-{\rm i}c')(\lambda_1-\lambda_2-{\rm i}c')(\lambda_3-\lambda_2+{\rm i}c')$$
$$-\,{\rm e}^{{\rm i}(x_1\lambda_2+x_2\lambda_1+x_3\lambda_3)}(\lambda_3-\lambda_1-{\rm i}c')(\lambda_3-\lambda_2-{\rm i}c')(\lambda_1-\lambda_2+{\rm i}c')\}, \quad \text{(A.10)}$$

Region VI ($x_2 < x_3 < x_1$)

$$\chi_{\rm VI}(x_1,x_2,x_3) = A\,{\rm e}^{\frac{-{\rm i}\pi\kappa}{2}}\{{\rm e}^{{\rm i}(x_1\lambda_1+x_2\lambda_2+x_3\lambda_3)}(\lambda_3-\lambda_2-{\rm i}c')(\lambda_3-\lambda_1+{\rm i}c')(\lambda_2-\lambda_1+{\rm i}c')$$
$$-\,{\rm e}^{{\rm i}(x_1\lambda_1+x_2\lambda_3+x_3\lambda_2)}(\lambda_2-\lambda_3-{\rm i}c')(\lambda_2-\lambda_1+{\rm i}c')(\lambda_3-\lambda_1+{\rm i}c')$$
$$+\,{\rm e}^{{\rm i}(x_1\lambda_3+x_2\lambda_1+x_3\lambda_2)}(\lambda_2-\lambda_1-{\rm i}c')(\lambda_2-\lambda_3+{\rm i}c')(\lambda_1-\lambda_3+{\rm i}c')$$
$$-\,{\rm e}^{{\rm i}(x_1\lambda_3+x_2\lambda_2+x_3\lambda_1)}(\lambda_1-\lambda_2-{\rm i}c')(\lambda_1-\lambda_3+{\rm i}c')(\lambda_2-\lambda_3+{\rm i}c')$$
$$+\,{\rm e}^{{\rm i}(x_1\lambda_2+x_2\lambda_3+x_3\lambda_1)}(\lambda_1-\lambda_3-{\rm i}c')(\lambda_1-\lambda_2+{\rm i}c')(\lambda_3-\lambda_2+{\rm i}c')$$
$$-\,{\rm e}^{{\rm i}(x_1\lambda_2+x_2\lambda_1+x_3\lambda_3)}(\lambda_3-\lambda_1-{\rm i}c')(\lambda_3-\lambda_2+{\rm i}c')(\lambda_1-\lambda_2+{\rm i}c')\}, \quad \text{(A.11)}$$

where

$$A = \frac{1}{\sqrt{6\prod_{j>k}[(\lambda_j-\lambda_k)^2+c'^2]}}. \quad \text{(A.12)}$$

As discussed above for the two particles, the periodic boundary conditions can be imposed in principle on any of the wavefunction arguments. Requiring x_1 to be periodic, $\chi(0,x_2,x_3) = \chi(L,x_2,x_3)$, gives

$$\chi_{\rm I}(0,x_2,x_3) = \chi_{\rm VI}(L,x_2,x_3), \qquad \text{for} \quad x_2 < x_3, \quad \text{(A.13)}$$

$$\chi_{\rm II}(0,x_2,x_3) = \chi_{\rm IV}(L,x_2,x_3), \qquad \text{for} \quad x_3 < x_2. \quad \text{(A.14)}$$

Except for the exchange-statistics phase factors, the wavefunctions in the six regions coincide with the wavefunctions of the Bose gas with the δ-function interaction of strength c' (25).

Therefore, the Bethe equations we obtain are the same as in the bosonic case with the only difference coming from the statistical phase factors. Conditions (A.13) and (A.14) produce six equations each, with only three of them being independent

$$\begin{aligned} e^{iL\lambda_1} &= e^{2i\pi\kappa}\left(\frac{\lambda_1-\lambda_2+ic'}{\lambda_1-\lambda_2-ic'}\right)\left(\frac{\lambda_1-\lambda_3+ic'}{\lambda_1-\lambda_3-ic'}\right),\\ e^{iL\lambda_2} &= e^{2i\pi\kappa}\left(\frac{\lambda_2-\lambda_1+ic'}{\lambda_2-\lambda_1-ic'}\right)\left(\frac{\lambda_2-\lambda_3+ic'}{\lambda_2-\lambda_3-ic'}\right),\\ e^{iL\lambda_3} &= e^{2i\pi\kappa}\left(\frac{\lambda_3-\lambda_1+ic'}{\lambda_3-\lambda_1-ic'}\right)\left(\frac{\lambda_3-\lambda_2+ic'}{\lambda_3-\lambda_2-ic'}\right). \end{aligned} \tag{A.15}$$

These equations can be written in the compact form similar to equation (26):

$$e^{iL\lambda_j} = e^{2i\pi\kappa}\prod_{k=1,k\neq j}^{3}\left(\frac{\lambda_j-\lambda_k+ic'}{\lambda_j-\lambda_k-ic'}\right). \tag{A.16}$$

If the periodic boundary conditions are imposed on the second variable, $\chi(x_1, 0, x_3) = \chi(x_1, L, x_3)$, i.e.,

$$\chi_{V}(0, x_2, x_3) = \chi_{II}(L, x_2, x_3), \qquad \text{for} \quad x_1 < x_3, \tag{A.17}$$

$$\chi_{VI}(x_1, 0, x_3) = \chi_{III}(x_1, L, x_3), \qquad \text{for} \quad x_1 > x_3, \tag{A.18}$$

we obtain either from (A.17) or (A.18) the following Bethe equations:

$$e^{iL\lambda_j} = \prod_{k=1,k\neq j}^{3}\left(\frac{\lambda_j-\lambda_k+ic'}{\lambda_j-\lambda_k-ic'}\right). \tag{A.19}$$

Finally, if we impose periodic boundary conditions on the third variable, $\chi(x_1, x_2, 0) = \chi(x_1, x_2, L)$, i.e.,

$$\chi_{III}(x_1, x_2, 0) = \chi_{I}(x_1, x_2, L), \qquad \text{for} \quad x_1 < x_2, \tag{A.20}$$

$$\chi_{IV}(x_1, x_2, 0) = \chi_{V}(x_1, x_2, L), \qquad \text{for} \quad x_2 < x_1, \tag{A.21}$$

the resulting Bethe equations are

$$e^{iL\lambda_j} = e^{-2i\pi\kappa}\prod_{k=1,k\neq j}^{3}\left(\frac{\lambda_j-\lambda_k+ic'}{\lambda_j-\lambda_k-ic'}\right). \tag{A.22}$$

The difference between the three forms of the Bethe equations (A.16), (A.19), (A.22) means that the periodic boundary conditions imposed on one variable automatically require the twisted boundary conditions on the other variables if one wants to keep the same Bethe equations. Similarly to the case of two particles, this can also be seen directly from the anyonic exchange symmetry (17) of the wavefunction. Suppose we set the periodic boundary conditions on the first variable:

$$\chi(0, x_2, x_3) = \chi(L, x_2, x_3). \tag{A.23}$$

Exchanging then the first two variables on both sides of equation (A.23) with the help of equation (17), we get the twisted boundary conditions for the second variable:

$$\chi(x_2, 0, x_3) = \chi(x_2, L, x_3)\, e^{-2i\pi\kappa}. \tag{A.24}$$

From (A.24), using again (17) we have

$$\chi(x_2, x_3, 0) = \chi(x_2, x_3, L)\, e^{-4i\pi\kappa}, \tag{A.25}$$

which are the twisted boundary conditions for the third variable which follow from the periodic conditions on the first. From any of the boundary conditions (A.23), (A.24), (A.25) we obtain the Bethe equations (A.16).

Similarly, periodic boundary conditions on the second variable give the following boundary conditions for the three-anyon wavefunction:

$$\begin{aligned}\chi(0,x_2,x_3) &= \chi(L,x_2,x_3)\,\mathrm{e}^{2\mathrm{i}\pi\kappa},\\ \chi(x_1,0,x_3) &= \chi(x_1,L,x_2),\\ \chi(x_2,x_3,0) &= \chi(x_2,x_3,L)\,\mathrm{e}^{-2\mathrm{i}\pi\kappa},\end{aligned} \tag{A.26}$$

and the Bethe equations (A.19). The same can be done starting with periodicity in the third variable. As in the case of two particles, we see that imposing periodic boundary conditions on the first and the last variables produces the Bethe equations, (A.16) and (A.22), which differ only by the sign of the statistical parameter κ. As discussed in detail for the two particles, this difference corresponds physically to different imbedding of the 1D loop of anyons into the underlying 2D system. In the two situations, the number of particles enclosed by the trajectories of successive particles x_j, $j = 1, 2, \ldots, N$, either increases from 0 to $(N-1)$ or decreases from $(N-1)$ to 0, as reflected in the corresponding boundary conditions of the multi-anyon wavefunction. In contrast to this, the requirement of periodicity of one of the 'internal' variables (e.g., x_2 in the case of three particles) produces the Bethe equations and boundary conditions, e.g. (A.19) and (A.26), that do not have this interpretation. They describe the situations with appropriate non-vanishing external phase shift $\phi \neq 0$, which twists uniformly the boundary conditions of all the variables. In the main text of our paper, we use the periodic boundary conditions with respect to the first variable of the anyonic wavefunction or introduce the external twist $\phi = \pi\kappa(N-1)$ which removes the anyonic shift of the quasiparticle momenta. As follows from the discussion in this appendix, the boundary conditions for the wavefunction of N anyons are given in these two situations by equations (23).

Appendix B. Particle–hole excitation

In this appendix, we find the energy and momentum of particle–hole excitations of the gas of anyons. As discussed in the main text, for twisted boundary conditions ($\beta = 1$), the ground state of anyons is equivalent to that of the Bose gas with periodic boundary conditions and coupling constant c', so the excitation energy and momentum coincide in this case with those known for the Bose gas (see chapter I.4 of [18]). For periodic boundary conditions ($\beta = 0$), the Bethe equations are the same as for the Bose gas with the boundary conditions twisted by the phase shift $2\pi\delta$, where $\delta = \{[\pi\kappa(N-1)]\}$. In the case of one hole with momentum λ_h and one particle with momentum λ_p the equations for the ground state and the excited state are

$$\text{Ground State, PBC:}\quad \lambda_j L + \sum_{k=1}^{N}\theta(\lambda_j-\lambda_k) = 2\pi\left(j-\frac{N+1}{2}\right)+2\pi\delta,\qquad j=1,\ldots,N, \tag{B.1}$$

$$\begin{aligned}\text{Excited State, PBC:}\quad &\tilde{\lambda}_j L + \sum_{k=1}^{N}\theta(\tilde{\lambda}_j-\tilde{\lambda}_k)+\theta(\tilde{\lambda}_j-\lambda_p)-\theta(\tilde{\lambda}_j-\lambda_h)\\ &= 2\pi\left(j-\frac{N+1}{2}\right)+2\pi\delta,\qquad j=1,\ldots,N.\end{aligned} \tag{B.2}$$

Comparing the equations for a particle–hole excitation in the case of twisted boundary conditions

$$\text{Ground State, TBC:}\quad \lambda_j^B L + \sum_{k=1}^{N} \theta\left(\lambda_j^B - \lambda_k^B\right) = 2\pi \left(j - \frac{N+1}{2}\right),\quad j = 1, \ldots, N, \tag{B.3}$$

$$\text{Excited State, TBC:}\quad \tilde{\lambda}_j^B L + \sum_{k=1}^{N} \theta\left(\tilde{\lambda}_j^B - \tilde{\lambda}_k^B\right) + \theta\left(\tilde{\lambda}_j^B - \lambda_p^B\right) - \theta\left(\tilde{\lambda}_j^B - \lambda_h^B\right) = 2\pi \left(j - \frac{N+1}{2}\right), \qquad j = 1, \ldots, N, \tag{B.4}$$

with (B.1) and (B.2), we find the following relations:

$$\lambda_j = \lambda_j^B + 2\pi\delta/L, \qquad \tilde{\lambda}_j = \tilde{\lambda}_j^B + 2\pi\delta/L, \quad (j = 1, \ldots, N) \tag{B.5}$$

$$\lambda_p = \lambda_p^B + 2\pi\delta/L, \qquad \lambda_h = \lambda_h^B + 2\pi\delta/L. \tag{B.6}$$

The energy and momentum of this excited state with respect to the ground state is $(\varepsilon_0(\lambda) = \lambda^2 - h)$,

$$\begin{aligned}\Delta E(\lambda_p, \lambda_h) &= \varepsilon_0(\lambda_p) - \varepsilon_0(\lambda_h) + \sum_{j=1}^{N} (\varepsilon_0(\tilde{\lambda}_j) - \varepsilon_0(\lambda_j)) \\ &= \varepsilon_0\left(\lambda_p^B\right) - \varepsilon_0\left(\lambda_h^B\right) + \sum_{j=1}^{N} \left(\varepsilon_0\left(\tilde{\lambda}_j^B\right) - \varepsilon_0\left(\lambda_j^B\right)\right) + 2\frac{2\pi\delta}{L} \\ &\quad \times \left(\lambda_p^B - \lambda_h^B + \sum_{j=1}^{N} \left(\tilde{\lambda}_j^B - \lambda_j^B\right)\right) \\ &= \Delta E^B\left(\lambda_p^B, \lambda_h^B\right) + 2\frac{2\pi\delta}{L} \Delta P^P\left(\lambda_h^B, \lambda_p^B\right),\end{aligned} \tag{B.7}$$

$$\Delta P(\lambda_p, \lambda_h) = \Delta P^B\left(\lambda_p^B, \lambda_h^B\right), \tag{B.8}$$

where $\Delta E^B\left(\lambda_p^B, \lambda_h^B\right)$ and $\Delta P^B\left(\lambda_p^B, \lambda_h^B\right)$ are the energy and momentum of a particle–hole excitation in the Bose gas with periodic boundary conditions, and λ_h^B and λ_p^B are given by (B.6).

From (B.7) we see that in the case of twisted boundary conditions, the Fermi velocity v_F^{TBC} will be the same as in the Bose gas, whereas for the periodic boundary conditions the Fermi velocity will be modified as

$$v_F^{\mathrm{PBC}} = v_F^{\mathrm{TBC}} + \frac{4\pi\delta}{L}. \tag{B.9}$$

References

[1] Leinaas J M and Myrheim J 1977 *Nuovo Cimento* B **37** 1
[2] Goldin G A, Menikoff R and Sharp D H 1980 *J. Math. Phys.* **21** 650

Goldin G A, Menikoff R and Sharp D H 1981 *J. Math. Phys.* **22** 1664
[3] Wilczek F 1982 *Phys. Rev. Lett.* **49** 957
[4] Arovas D, Schrieffer J R and Wilczek F 1984 *Phys. Rev. Lett.* **53** 722
[5] Goldman V J and Su B 1995 *Science* **267** 1010
[6] Averin D V and Nesteroff J A 2007 *Phys. Rev. Lett.* **99** (*Preprint* 0704.0439)
[7] Averin D V and Goldman V J 2002 *Solid State Commun.* **121** 25
[8] Das Sarma S, Freedman M and Nayak C 2005 *Phys. Rev. Lett.* **94** 166802
[9] Kitaev A 2003 *Ann. Phys.* **303** 2
[10] Rabello S J B 1996 *Phys. Rev. Lett.* **76** 4007
[11] Aglietti U, Griguolo L, Jackiw R, Pi S Y and Seminara D 1996 *Phys. Rev. Lett.* **77** 4406
[12] Haldane F D M 1991 *Phys. Rev. Lett.* **67** 937
[13] Kundu A 1999 *Phys. Rev. Lett.* **83** 1275
[14] Batchelor M T, Guan X-W and Oelkers N 2006 *Phys. Rev. Lett.* **96** 210402 (*Preprint* cond-mat/0603643)
[15] Batchelor M T and Guan X-W 2006 *Phys. Rev.* B **74** 195121 (*Preprint* cond-mat/0606353)
[16] Batchelor M T, Guan X-W and He J-S 2007 *J. Stat. Mech.* P03007 (*Preprint* cond-mat/0611450)
[17] Lieb E H and Liniger W 1963 *Phys. Rev.* **130** 1605
[18] Korepin V E, Bogoliubov N M and Izergin A G 1993 *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge: Cambridge University Press)
[19] Calabrese P and Mintchev M 2007 *Preprint* cond-mat/0703117
[20] Liguori A, Mintchev M and Pilo L 2000 *Nucl. Phys.* B **569** 577
[21] Ilieva N and Thirring W 1999 *Eur. Phys. J.* C **6** 705
Ilieva N and Thirring W 1999 *Theor. Mat. Phys.* **121** 1294
[22] Girardeau M D 2006 *Phys. Rev. Lett.* **97** 100402
[23] McGuire J B 1964 *J. Math. Phys.* **5** 622
[24] Bogoliubov N M, Izergin A G and Reshetikhin N Yu 1986 *JETP Lett.* **44** 521
[25] Bogoliubov N M, Izergin A G and Reshetikhin N Yu 1987 *J. Phys. A: Math. Gen.* **20** 5361
[26] Berkovich A and Murthy G 1988 *J. Phys. A: Math. Gen.* **21** L395
[27] Berkovich A and Murthy G 1988 *J. Phys. A: Math. Gen.* **21** 3703
[28] Woynarovich F, Eckle H P and Truong T T 1989 *J. Phys. A: Math. Gen.* **22** 4027
[29] Belavin A A, Polyakov A M and Zamolodchikov A B 1984 *Nucl. Phys.* B **241** 333
[30] Itzykson C, Saleur M and Zuber J B 1986 *Conformal Invariance and Applications to Statistical Mechanics* (Singapore: World Scientific)
[31] Cardy J L 1989 Lectures given at Les Houches, Session XLIX, 1988 *Champs, Cordes et Phénomènes Critiques: Fields, Strings and Critical Phenomena* ed E Brezin and J Zinn-Justin (Amsterdam: Elsevier)
[32] Ginsparg P 1989 Lectures given at Les Houches, Session XLIX, 1988 *Champs, Cordes et Phénomènes Critiques: Fields, Strings and Critical Phenomena* ed E Brezin and J Zinn-Justin (Amsterdam: Elsevier)
[33] Blöte H W, Cardy J L and Nightingale M P 1986 *Phys. Rev. Lett.* **56** 742
[34] Affleck I 1986 *Phys. Rev. Lett.* **56** 746
[35] Friedan D, Qui Z and Shenker S 1984 *Phys. Rev. Lett.* **52** 1575
[36] Dijkgraaf R, Verlinde E and Verlinde H 1988 *Comm. Math. Phys.* **115** 649
[37] Cardy J L 1986 *Nucl. Phys.* B **270** 186
[38] Haldane F D M 1981 *Phys. Rev. Lett.* **97** 1840
Haldane F D M 1981 *J. Phys. C: Solid State Phys.* **15** 2585
[39] Cazalilla M A 2004 *J. Phys. B: At. Mol. Opt. Phys.* **37** S1
[40] Santachiara R, Stauffer F and Cabra D C 2006 *Preprint* cond-mat/0610402
[41] Lenard A 1964 *J. Math. Phys.* **5** 930